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INVARIANTS OF THE FUNCTION $F(x, y, x', y')$ IN THE CALCULUS OF VARIATIONS*

BY

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Introduction.

In KNESER's *Lehrbuch der Variationsrechnung*, § 16, and in BOLZA's *Lectures on the Calculus of Variations*, § 35, the transformation of the definite integral

$$I = \int_{t_0}^{t_1} F(x, y, x', y')\,dt$$

by a "point transformation,"

$$x = X(u, v), \qquad y = Y(u, v),$$

is incidentally considered, and the invariance of the expressions

$$F_1(x, y, x', y') = \frac{F_{x'x'}}{y'^2} = \frac{F_{x'y'}}{-x'y'} = \frac{F_{y'y'}}{x'^2},$$

$$T(x, y, x', y', x'', y'') = F_{xy'} - F_{yx'} + F_1(x'y'' - x''y'),$$

$$E(x, y, x', y', x^{\backprime}, y^{\backprime}) = F(x, y, x^{\backprime}, y^{\backprime}) - [x^{\backprime} F_{x'}(x, y, x', y') + y^{\backprime} F_{y'}(x, y, x', y')],$$

is proved. The object of the present paper is to study this transformation more in detail, and to derive further invariants which are of importance for the Calculus of Variations.†

In Chapter I the general definitions concerning invariants of the function $F(x, y, x', y')$ with respect to point transformations are given, and a process (which does not essentially differ from the "δ-process" of the Calculus of Variations) is developed, which transforms an absolute invariant again into an absolute invariant. This permits us to derive from known invariants new invariants.

In Chapter II, after a short discussion of the invariants arising out of the first variation, a new absolute invariant

$$K(x, y, x', y', x'', y'', x''', y''')$$

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† See also a paper by G. LANDSBERG, *Krümmungstheorie und Variationsrechnung*, Jahresbericht der Deutschen Mathematiker-Vereinigung, vol. 16, 1907.

connected with the second variation is obtained by means of the process described in Chapter I. When the curve for which the invariant K is computed is in particular an extremal, the invariant K takes the simple form

$$\overline{K}=\frac{1}{4}\frac{F_1'^2}{F_1^2}-\frac{1}{2}\frac{F_1''}{F_1}-\frac{F_2}{F_1}.$$

In Chapter III a transformation of the parameter t of the curve is combined with the point transformation, and by a proper modification of the invariant K a function

$$K_0(x, y, x', y', x'', y'', x''', y''')$$

is obtained which remains invariant not only under every point transformation but at the same time under every parameter transformation. For the case of an extremal the expression K_0 reduces to

$$\overline{K}_0=\frac{1}{F^2}\left[\overline{K}-\frac{1}{2}\left(\frac{F''}{F}-\frac{3}{2}\frac{F'^2}{F^2}\right)\right].$$

This leads to an invariantive normal form of the second variation, viz.

$$\delta^2 I=\frac{\epsilon^2}{2}\int_{\alpha_0}^{\alpha_1}\left[\left(\frac{dV}{d\alpha}\right)^2-\overline{K}_0 V^2\right]d\alpha.$$

In Chapter IV these results are applied to the case of the geodesics, and it is shown that the invariant $\overline{K}_0$ is in this case identical with the Gaussian curvature K; while the general invariant K_0 is expressible in terms of $1/\rho$, the Gaussian curvature of the surface, and of $1/\rho_g$, the geodesic curvature of the curve for which the value of K_0 is taken, by means of the formula

$$K_0=\frac{1}{\rho}-\frac{1}{2\rho_g^2}.$$

Finally, the result concerning the second variation is applied to JACOBI's theorem on the conjugate points of geodesics on surfaces of negative curvature.

CHAPTER I.* *General theory of the invariants of the calculus of variations for the simplest type of problems.*

§ 1. *Extended point transformations.*

We consider a point transformation connecting two systems of rectangular coördinates,

(1) $$x = X(u, v), \qquad y = Y(u, v).$$

* This chapter is based upon a series of lectures given by Professor BOLZA in his Seminar in the Calculus of Variations during the Spring Quarter, 1905, at the University of Chicago.

For simplicity we suppose that the functions $X(u, v)$, $Y(u, v)$ are analytic functions of u, v, regular in the domain under consideration. Further we assume that the Jacobian

$$D(u, v) \equiv \frac{\partial(X, Y)}{\partial(u, v)}$$

is different from zero in this domain, thus insuring a one-to-one correspondence between every sufficiently small part of this domain and its image in the xy-plane.* The inverse of the transformation (1) we denote by

$$(1a) \qquad u = U(x, y), \qquad v = V(x, y).$$

We consider now a curve in the uv-plane

$$L: \qquad u = \phi(t), \qquad v = \psi(t),$$

which we suppose to be regular † in a certain interval (t_0, t_1). Let its image in the xy-plane be

$$(2) \qquad L': \quad \begin{cases} x = X[\phi(t), \psi(t)] = \Phi(t), \\ y = Y[\phi(t), \psi(t)] = \Psi(t). \end{cases}$$

Differentiating (2) with respect to t, and denoting derivatives with respect to t by accents, we obtain

$$(3) \qquad x' = X_u u' + X_v v', \qquad y' = Y_u u' + Y_v v',$$

$$(4) \qquad \begin{aligned} x'' &= X_{uu} u'^2 + 2X_{uv} u'v' + X_{vv} v'^2 + X_u u'' + X_v v'', \\ y'' &= Y_{uu} u'^2 + 2Y_{uv} u'v' + Y_{vv} v'^2 + Y_u u'' + Y_v v'', \end{aligned}$$

and so on.

The totality of all point transformations (1) form a group, T, which in LIE's terminology is an infinite, ‡ continuous group. § If we combine (1) and (3) and consider u', v', x', y', not as derivatives of certain functions with respect to t, but as new variables, the combination (1) and (3) represents an "extended point-transformation" || between the variables, u, v, u', v', on the one hand, and the variables x, y, x', y', on the other hand. It is easily proved that the totality of transformations (1), (3) forms a group, which we denote by T'. Similarly

* OSGOOD, *Lehrbuch der Functionentheorie*, vol. 1, p. 56.

† I. e., $\phi(t)$, $\psi(t)$ are analytic functions of t, regular in (t_0, t_1), and $\phi'(t)$, $\psi'(t)$ do not vanish simultaneously in (t_0, t_1). Compare KNESER, *Lehrbuch der Variations-Rechnung*, p. 3.

‡ "Infinite" inasmuch as the transformation (1) contains the arbitrary functions $X(u, v)$, $Y(u, v)$.

§ Compare LIE, *Die Grundlagen für die Theorie der unendlichen continuirlichen Gruppen*, Leipziger Berichte, 1891, p. 316, and LIE-SCHEFFERS, *Continuirliche Gruppen*, p. 764.

|| Compare LIE-SCHEFFERS, *Geometrie der Berührungs-Transformationen*, p. 12. It must, however, be remarked that LIE assumes the curve in the non-parametric form $v = f(u)$, and accordingly obtains (3) in non-homogeneous form.

by adjoining further the equations (4) to T', we would obtain a "twice extended point transformation," and a corresponding group T'', and so on.

§ 2. *Definition of invariants of the function* $F(x, y, x', y')$.

We now apply the group T' to a function $F(x, y, x', y')$. We suppose that F is an analytic function of its four arguments, regular in the vicinity of every point

$$x = a, \qquad y = b, \qquad x' = a', \qquad y' = b',$$

for which (a, b) lies in a certain region R of the xy-plane, while at the same time $(a', b') \neq (0, 0)$. We suppose further that F satisfies the homogeneity condition of the calculus of variations,* namely,

$$F(x, y, \kappa x', \kappa y') = \kappa F(x, y, x', y')$$

for every positive κ. Substituting for x, y, x', y' their values from (1), (3) in our function $F(x, y, x', y')$, we obtain

$$F(x, y, x', y') = F(X, Y, X_u u' + X_v v', Y_u u' + Y_v v'),$$

or

$$F(x, y, x', y') = G(u, v, u', v'), \tag{5}$$

where G is defined by

$$G(u, v, u', v') = F(X, Y, X_u u' + X_v v', Y_u u' + Y_v v').$$

From this and the homogeneity condition, it follows at once that

$$G(u, v, \kappa u', \kappa v') = \kappa G(u, v, u', v')$$

for every positive κ.

From (5) we can compute the partial derivatives of G, for instance,

$$G_{u'} = F_{x'} X_u + F_{y'} Y_u, \qquad G_{v'} = F_{x'} X_v + F_{y'} Y_v, \tag{6}$$

whence

$$F_{x'} = \frac{1}{D}(Y_v G_{u'} - Y_u G_{v'}), \qquad F_{y'} = \frac{1}{D}(-X_v G_{u'} + X_u G_{v'}). \tag{7}$$

Following the general method outlined by LIE in his paper *Ueber Differential-Invarianten,*† we now adjoin equations (7) to (1) and (3), considering $F_{x'}$, $F_{y'}$, $G_{u'}$, $G_{v'}$ as new variables. We obtain then a transformation between the variables x, y, x', y', $F_{x'}$, $F_{y'}$, on the one hand, and u, v, u', v', $G_{u'}$, $G_{v'}$, on the other. The totality of these transformations corresponding to the totality of point transformations (1) form a group T_1'.

*Compare KNESER, *Lehrbuch der Variationsrechnung*, § 3; BOLZA, *Lectures on the Calculus of Variations*, § 24 *b*.

†Mathematische Annalen, vol. 24 (1884), p. 537 ff., and in particular p. 569.

We might also extend T_1' by adjoining the solutions F_x, F_y of the equations

$$G_u = F_x X_u + F_y Y_u + F_{x'}(X_{uu} u' + X_{uv} v') + F_{y'}(Y_{uu} u' + Y_{uv} v'),$$
$$G_v = F_x X_v + F_y Y_v + F_{x'}(X_{uv} u' + X_{vv} v') + F_{y'}(Y_{uv} u' + Y_{vv} v'),$$

and still further by adjoining equations for the second partial derivatives of F and G, and so on. In this manner we should finally get an infinite continuous group $T_m^{(\mu)}$, whose transformations connect x, y, x', y', $\cdots$, $x^{(\mu)}$, $y^{(\mu)}$, F_x, F_y, $F_{x'}$, $F_{y'}$, $\cdots$, $F_{y'^{(m)}}$ with u, v, u', v', $\cdots$, $u^{(\mu)}$, $v^{(\mu)}$, G_u, G_v, $G_{u'}$, $G_{v'}$, $\cdots$, $G_{v'^{(m)}}$ where m is the order of the highest partial derivative of F and G.

We next define *Invariants* under these infinite continuous groups as follows. Let $I_F(x, y, x', y', x'', y'', \cdots, x^{(\mu)}, y^{(\mu)})$ be a function of the arguments indicated, and of F and its partial derivatives up to those of the m-th order. Further let $I_G(u, v, u', v', u'', v'', \cdots)$ be the *same* function of $u, v, u', v', u'', v'', \cdots, u^{(\mu)}, v^{(\mu)}$, and of G and its partial derivatives up to those of the m-th order. Then if $I_G(u, v, u', v', \cdots) = I_F(x, y, x', y', \cdots)$ for every transformation of the group $T_m^{(\mu)}$, we say I_F is an *absolute Invariant* * under $T_m^{(\mu)}$. We call I_F an *Invariant of Index* ρ if $I_G(u, v, u', v', \cdots) = D^\rho I_F(x, y, x', y', \cdots)$, where

$$D = \frac{\partial(X, Y)}{\partial(u, v)}.$$

Furthermore call μ, which is the highest order of the derivatives with respect to t occurring in I_F, *the order of the invariant*, and the highest order, m, of the partial derivatives of F, which occurs in I_F, *the class of the invariant.* As examples, we mention

$$G_1(u, v, u', v') = D^2 F_1(x, y, x', y')\dagger \tag{8}$$

of class 2, order 1, index 2, and

$$T_G(u, v, u', v', u'', v'') = D T_F(x, y, x', y', x'', y'') \tag{9}$$

of class 2, order 2, index 1, where

$$F_1 = \frac{F_{x'x'}}{y'^2} = -\frac{F_{x'y'}}{x'y'} = \frac{F_{y'y'}}{x'^2}, \qquad G_1 = \frac{G_{u'u'}}{v'^2} = -\frac{G_{u'v'}}{u'v'} = \frac{G_{v'v'}}{u'^2},$$

$$T_F = F_{xy'} - F_{yx'} + F_1(x'y'' - x''y'), \qquad T_G = G_{uv'} - G_{vu'} + G_1(u'v'' - u''v'),$$

thus showing the invariance of the so-called LEGENDRE and EULER conditions of the calculus of variations under extended point transformations.

* "Differential Invariant" in LIE's terminology.

† BOLZA, loc. cit., p. 183.

§ 3. *Invariants with several sets of cogredient variables.*

We now introduce a second series of variables

$$x^{\backprime},\ y^{\backprime},\ x^{\backprime\backprime},\ y^{\backprime\backprime},\ \cdots, \qquad u^{\backprime},\ v^{\backprime},\ u^{\backprime\backprime},\ v^{\backprime\backprime},\ \cdots$$

cogredient with

$$x',\ y',\ x'',\ y'',\ \cdots, \qquad u',\ v',\ u'',\ v'',\ \cdots,$$

that is, connected by the same transformations,

$$x^{\backprime} = X_u u^{\backprime} + X_v v^{\backprime}, \qquad y^{\backprime} = Y_u u^{\backprime} + Y_v v^{\backprime}, \tag{10}$$

and so on. We thus obtain a further extension of our group and corresponding invariants of F, with two sets of cogredient variables, satisfying for all transformations of the group the equation,

$$I_G(u,\ v,\ u',\ v',\ \cdots,\ u^{\backprime},\ v^{\backprime},\ \cdots) = D^\rho I_F(x,\ y,\ x',\ y',\ \cdots,\ x^{\backprime},\ y^{\backprime},\ \cdots).$$

An example* of such an invariant is

$$x^{\backprime} F_{x'}(x,\ y,\ x',\ y') + y^{\backprime} F_{y'}(x,\ y,\ x',\ y'),$$

for which

$$u^{\backprime} G_{u'}(u,\ v,\ u',\ v') + v^{\backprime} G_{v'}(u,\ v,\ u',\ v') = x^{\backprime} F_{x'}(x,\ y,\ x',\ y') + y^{\backprime} F_{y'}(x,\ y,\ x',\ y'), \tag{11}$$

as may be verified at once by means of (6) and (10).

From (11) it follows at once that the WEIERSTRASS E-function

$$E_F(x,\ y,\ x',\ y',\ x^{\backprime},\ y^{\backprime}) = F(x,\ y,\ x^{\backprime},\ y^{\backprime}) - [x^{\backprime} F_x(x,\ y,\ x',\ y') + y^{\backprime} F_{y'}(x,\ y,\ x',\ y')]$$

is an absolute invariant, i. e.,

$$E_G(u,\ v,\ u',\ v',\ u^{\backprime},\ v^{\backprime}) = E_F(x,\ y,\ x',\ y',\ x^{\backprime},\ y^{\backprime}).$$

Invariants of this kind also occur when we consider two curves,

$$L: \qquad x = x(t), \qquad y = y(t),$$

and

$$\tilde{L}: \qquad x = \tilde{x}(\tau), \qquad y = \tilde{y}(\tau).$$

At the point of intersection P we have

$$x = \tilde{x}, \qquad y = \tilde{y},$$

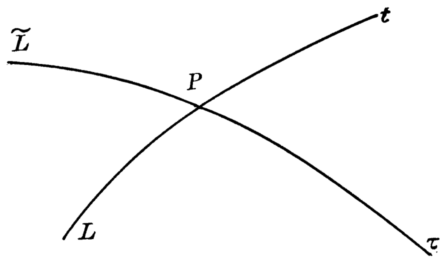
and if we put at P

$$x' = \frac{dx}{dt}, \qquad y' = \frac{dy}{dt}, \qquad x^{\backprime} = \frac{d\tilde{x}}{d\tau}, \qquad y^{\backprime} = \frac{d\tilde{y}}{d\tau},$$

equation (11) shows that the condition of transversality of L and $\tilde{L}$ is preserved in passing from the xy-plane to the uv-plane.

Another invariant of this kind is the expression $x'y^{\backprime} - x^{\backprime}y'$, for which

* BOLZA, loc. cit., p. 183, eq. 28.



(12) $$(u'v^{\backprime} - u^{\backprime}v') = D^{-1}(x'y^{\backprime} - x^{\backprime}y').$$

This last example has an important application in the calculus of variations. When we consider a set of curves in the uv-plane

$$u = u(t, a), \qquad v = v(t, a),$$

then the corresponding set in the xy-plane is

(13) $$x = x(t, a), \qquad y = y(t, a),$$

where

$$x(t, a) = X[u(t, a), v(t, a)], \qquad y(t, a) = Y[u(t, a), v(t, a)].$$

Hence,

$$\frac{\partial x}{\partial t} = \frac{\partial X}{\partial u}\frac{\partial u}{\partial t} + \frac{\partial X}{\partial v}\frac{\partial v}{\partial t}, \qquad \frac{\partial x}{\partial a} = \frac{\partial X}{\partial u}\frac{\partial u}{\partial a} + \frac{\partial X}{\partial v}\frac{\partial v}{\partial a},$$

$$\frac{\partial y}{\partial t} = \frac{\partial Y}{\partial u}\frac{\partial u}{\partial t} + \frac{\partial Y}{\partial v}\frac{\partial v}{\partial t}, \qquad \frac{\partial y}{\partial a} = \frac{\partial Y}{\partial u}\frac{\partial u}{\partial a} + \frac{\partial Y}{\partial v}\frac{\partial v}{\partial a}.$$

The variables

$$x' = \frac{\partial x}{\partial t}, \qquad y' = \frac{\partial y}{\partial t}, \qquad u' = \frac{\partial u}{\partial t}, \qquad v' = \frac{\partial v}{\partial t},$$

are therefore cogredient with

$$x^{\backprime} = \frac{\partial x}{\partial a}, \qquad y^{\backprime} = \frac{\partial y}{\partial a}, \qquad u^{\backprime} = \frac{\partial u}{\partial a}, \qquad v^{\backprime} = \frac{\partial v}{\partial a},$$

and consequently from (12) the *Jacobian*

$$\Delta(t, a) = \frac{\partial(x, y)}{\partial(t, a)}$$

is an *invariant of index* -1; i. e.,

$$\frac{\partial(u, v)}{\partial(t, a)} = D^{-1}\frac{\partial(x, y)}{\partial(t, a)}.$$

If, in particular, the set of curves (13) is the set of extremals through a fixed point P_0, then the equation

$$\Delta(t, a_0) = 0$$

furnishes the conjugate * point P_0' to P_0 on the particular extremal $a = a_0$. But on account of (9) the image of the extremals in the uv-plane is the set of extremals through the image Q_0 of P_0.

It follows, since Δ is invariant, *that the conjugate Q_0' of Q_0 on the extremal $a = a_0$ in the uv-plane, is the image of the point P_0'. In other words, the conjugate of the image is the image of the conjugate.*†

Another result from (12) is as follows: If $I_F(x, y, x', y', \cdots)$ be an invariant of index ρ, then

$$(x'y^{\backprime} - x^{\backprime}y')^{\rho} I_F(x, y, x', y' \cdots)$$

is an *absolute invariant.* For instance, $F_1(x'y^{\backprime} - x^{\backprime}y')^2$ is an absolute invariant, according to (8).

The following lemma will be of importance:

From every invariant which contains only the variables $x^{\backprime}$, $y^{\backprime}$ of the second set, and which is, moreover, homogeneous in $x^{\backprime}$, $y^{\backprime}$, an invariant with only one set of variables can be derived by replacing $x^{\backprime}$, $y^{\backprime}$ by $F_{y'}$, $-F_{x'}$, respectively.

Proof. From (7) we have

$$D(F_{y'}) = (G_{v'})X_u + (-G_{u'})X_v, \qquad D(-F_{x'}) = (G_{v'})Y_u + (-G_{u'})Y_v.$$

Hence the equations (10) are satisfied by

$$u^{\backprime} = G_{v'}, \qquad v^{\backprime} = -G_{u'}, \qquad x^{\backprime} = DF_{y'}, \qquad y^{\backprime} = -DF_{x'}.$$

If, therefore,

$$I_G(u, v, u', v', \cdots, u^{\backprime}, v^{\backprime}) = D^{\rho} I_F(x, y, x', y', \cdots, x^{\backprime}, y^{\backprime}),$$

for all transformations of the group under consideration, then we have in particular,

$$I_G(u, v, u', v', \cdots, G_{v'}, -G_{u'}) = D^{\rho} I_F(x, y, x', y', \cdots, DF_{y'}, -DF_{x'}).$$

If, furthermore, I is homogeneous in $x^{\backprime}$, $y^{\backprime}$ of degree m, then

$$I_G(u, v, u', v', \cdots, G_{v'}, -G_{u'}) = D^{\rho+m} I_F(x, y, x', y', \cdots, F_{y'}, -F_{x'}),$$

and I is an invariant with only one set of variables. The index is increased by the degree m of the homogeneity of I in $x^{\backprime}$, $y^{\backprime}$.

§ 4. *Methods for the construction of invariants.*

For the determination of all invariants of a given group $T^{(\mu)}_{(m)}$ one might use the general method developed by LIE in his paper *Ueber Differential-Invari-*

* BOLZA, loc. cit., p. 63.

† First given by A. L. UNDERHILL in a paper before the Chicago Section, April, 1905.

anten,* which reduces the problem to the solution of a complete system of linear partial differential equations.† The application of this method to the present problem becomes, however, very complicated on account of the great number of adjoined variables. We therefore employ different methods, which lead more directly to those invariants which are of paramount importance in the calculus of variations.

The simplest of these invariants have been used already as examples in the preceding sections. They are the functions

$$F(x, y, x', y'), \quad x'F_{x'}(x, y, x', y') + y'F_{y'}(x, y, x', y'), \quad F_1(x, y, x', y'),$$
$$T(x, y, x', y', x'', y''), \quad E(x, y, x', y', x', y'), \quad \Delta(t, a).$$

In order to obtain further invariants, we now develop methods for the derivation of invariants from already known invariants. We denote by $I_F(x, y, x', y', \cdots)$ an absolute invariant under $T^{(\mu)}_{(m)}$, so that

$$I_G(u, v, u', v', \cdots) = I_F(x, y, x', y', \cdots) \tag{14}$$

for every transformation of $T^{(\mu)}_{(m)}$.

a) The method of differentiation.

Consider now u, v as functions of t, and $u^{(k)}, v^{(k)}$ as their k-th derivatives with respect to t. Then x, y will be functions of t determined by (2), and similarly their derivatives $x', y', x'', y'', \cdots$ by (3), (4) and the successive derived equations. The equation (14) becomes then an identity in t, and it may be differentiated with respect to t. We have

$$\frac{dI_G}{dt} = \frac{dI_F}{dt}, \tag{15}$$

i. e.,

$$\begin{aligned} &\frac{\partial I_G}{\partial u}u' + \frac{\partial I_G}{\partial v}v' + \frac{\partial I_G}{\partial u'}u'' + \frac{\partial I_G}{\partial v'}v'' + \cdots \\ &\qquad = \frac{\partial I_F}{\partial x}x' + \frac{\partial I_F}{\partial y}y' + \frac{\partial I_F}{\partial x'}x'' + \frac{\partial I_F}{\partial y'}y'' + \cdots, \end{aligned} \tag{16}$$

where it will be remembered that I_F is a function of $x, y, x', y', \cdots$ and of F and some of its partial derivatives which are themselves functions of x, y, x, y'. Equation (16) appears at first as an identity in t. But it holds for *any* functions u, v having derivatives of sufficiently high order. Let $u_0, v_0, u_0', v_0', u_0'', v_0'', \cdots$ be an arbitrary system of values of the variables $u, v, u', v', u'', v'', \cdots$, and $x_0, y_0, x_0', y_0', x_0'', y_0'', \cdots$ the corresponding values of $x, y, x', y', x'', y'', \cdots$ ob-

*Mathematische Annalen, vol. 24 (1884), p. 537.

†This method was used by ZORAWSKI for the geodesic problem; Acta Mathematica, vol. 16 (1892), p. 1.

tained by the transformations (1), (3), (4), and so on. Then in order to show that (16) holds for this arbitrarily chosen system of values, we choose for the functions $u(t)$, $v(t)$ which we substitute in (15),

$$u(t) = u_0 + u_0' \frac{t}{1!} + u_0'' \frac{t^2}{2!} + \cdots + u_0^{(\mu)} \frac{t^{(\mu)}}{\mu!},$$

$$v(t) = v_0 + v_0' \frac{t}{1!} + v_0'' \frac{t^2}{2!} + \cdots + v_0^{(\mu)} \frac{t^{(\mu)}}{\mu!},$$

and put after the differentiation $t = 0$. The right side of (16) is therefore a *new absolute invariant*, whose class and order are each one unit higher than that of I_F, and it accordingly corresponds to a further extended group. Thus we obtain from the absolute invariant,

$$I = F = x'F_{x'} + y'F_{y'}*$$

of order 1 and class 1, the new absolute invariant

$$x'F_x + y'F_y + x''F_{x'} + y''F_{y'}$$

which is of order 2 and class 2.

b) The δ-process.

We suppose now that the functions $u(t)$, $v(t)$ considered under (a) depend upon a parameter ϵ. Then equation (14) is an identity in t and ϵ, and we may therefore differentiate it with respect to ϵ,

$$\frac{\partial I_G}{\partial \epsilon} = \frac{\partial I_F}{\partial \epsilon},$$

i. e., if we denote differentiation with respect to ϵ by a dot,

$$(17)\quad \frac{\partial I_G}{\partial u}\dot{u} + \frac{\partial I_G}{\partial v}\dot{v} + \frac{\partial I_G}{\partial u'}\dot{u}' + \frac{\partial I_G \dot{v}'}{\partial v'} + \cdots = \frac{\partial I_F}{\partial x}\dot{x} + \frac{\partial I_F}{\partial v}\dot{v} + \frac{\partial I_F}{\partial x'}\dot{x}' + \frac{\partial I_F}{\partial y'}\dot{y}' + \cdots.$$

This equation appears again at first as an identity in t, ϵ. But since $u(t, \epsilon)$, $v(t, \epsilon)$ are arbitrary, (17) remains true for all systems of values of

$$u,\ v,\ u',\ v',\ \cdots, \qquad \dot{u},\ \dot{v},\ \dot{u}',\ \dot{v}',\ \cdots,$$

on the one hand, and of

$$x,\ y,\ x',\ y',\ \cdots, \qquad \dot{x},\ \dot{y},\ \dot{x}',\ \dot{y}',\ \cdots,$$

on the other hand, which are connected by (1), (3), (4) and so on, and also by the following relations obtained from (1), (2), (4) by differentiation with respect to ϵ:

* BOLZA, *Lectures*, p. 120, eq. 9.

$$\dot{x} = X_u\dot{u} + X_v\dot{v}, \qquad \dot{y} = Y_u\dot{u} + Y_v\dot{v},$$

$$\dot{x}' = X_{uu}\dot{u}u' + X_{uv}(\dot{u}v' + u'\dot{v}) + X_{vv}\dot{v}v' + X_u\dot{u}' + X_v\dot{v}',$$

$$\dot{y}' = Y_{uu}\dot{u}u' + Y_{uv}(\dot{u}v' + u'\dot{v}) + Y_{vv}\dot{v}v' + Y_u\dot{u}' + Y_v\dot{v}'.$$

The proof of this last statement is entirely analogous to the corresponding proof given under (a). The right hand side of (17) is therefore an absolute invariant under a further extended group obtained by means of these last equations by adjoining the new variables $\dot{x}, \dot{y}, \dot{x}', \dot{y}', \cdots$.

Our process transforms therefore an absolute invariant into an absolute invariant, under a further extended group. We shall call this process the "δ-process," because it is essentially identical with the δ-process of the calculus of variations, * and we shall use, in the sequel, the symbol δ to indicate the first order term of an expansion with respect to a parameter ϵ. Thus

$$\delta I_F = \dot{I}_F\epsilon = \frac{\partial I_F}{\partial\epsilon}\epsilon.$$

Similarly

$$\delta x = \dot{x}\epsilon = \frac{\partial x}{\partial \epsilon}\epsilon, \qquad \delta x' = \dot{x}'\epsilon = \frac{\partial^2 x}{\partial\epsilon\,\partial t}\epsilon,$$

and so on.

If we have found by the "δ-process" an absolute invariant with respect to the group extended by the adjunction of

$$\dot{x}, \dot{y}, \dot{x}', \dot{y}', \cdots,$$

we may apply the process a second time, and thus obtain a new absolute invariant with respect to the group obtained by adjoining the higher derivatives with respect to ϵ,

$$\ddot{x}, \ddot{y}, \ddot{x}', \ddot{y}', \cdots.$$

Numerous examples of this process will occur in the next chapter.

Chapter II. *Invariants under extended point transformations.*

§ 5. *The invariants arising from the first variation.*

We apply the general principles of Chapter I to the invariants connected with the first and second variations of

$$I = \int_{t_0}^{t_1} F(x, y, x', y')\,dt.$$

The starting point is the fact that F is an absolute invariant (cf. (5)) for the

* Bolza, *Lectures*, § 4c. If we put after the differentiation $\varepsilon = 0$, and then multiply by ε, we obtain exactly the δ-process of the calculus of variations.

group of extended transformations (1), (3); whence it follows by § 4 that δF and $\delta^2 F$ are also absolute invariants under the same group. We shall be able to break up each of these into invariants which are directly connected with the minimizing of I.

The first variation of F may be thrown into the two forms *

$$\delta F = \delta x\left(F_x - \frac{d}{dt}F_{x'}\right) + \delta y\left(F_y - \frac{d}{dt}F_{y'}\right) + \frac{d}{dt}(F_{x'}\delta x + F_{y'}\delta y)$$

$$= Tw + \frac{d}{dt}(F_{x'}\delta x + F_{y'}\delta y),$$

where

$$w = y'\delta x - x'\delta y.$$

Since δx and δy are cogredient with x' and y', it follows from (11) that

$$F_{x'}\delta x + F_{y'}\delta y$$

is an absolute invariant; and also, by § $4a$, its derivative with respect to t. Hence it follows, since δF is an absolute invariant, that Tw must also be an absolute invariant under the extended group of transformations (1), (3). We define $\delta_1 F$ and $\delta_2 F$ by the equations

$$(18)\qquad \begin{aligned} \delta_1 F &= \delta x\left(F_x - \frac{d}{dt}F_{x'}\right) + \delta y\left(F_y - \frac{d}{dt}F_{y'}\right) = Tw, \\ \delta_2 F &= \delta x F_{x'} + \delta y F_{y'}, \end{aligned}$$

$\delta_1 F$ and $\delta_2 F$ each being an absolute invariant.

§ 6. *First form of* $\delta(\delta_1 F)$.

The work will be similar to the WEIERSTRASS transformation of the second variation,† but will be more general inasmuch as we do not assume, as WEIERSTRASS does, that the second variation is computed along an extremal. We define

$$L = F_{xx'} - y'y''F_1, \quad M = \tfrac{1}{2}[F_{xy'} + F_{yx'} + F_1(x'y'' + x''y')], \quad N = F_{yy'} - x'x''F_1,$$

whence it follows that

$$Lx' + My' = F_x - \tfrac{1}{2}Ty', \qquad Mx' + Ny' = F_y + \tfrac{1}{2}Tx'.$$

With WEIERSTRASS we set

$$L_1 = F_{xx} - \frac{dL}{dt} - y''^2 F_1,$$

* BOLZA, loc. cit., § $25a$.

† BOLZA, loc. cit., § $27a$.

$$(19) \qquad M_1 = F_{xy} - \frac{dM}{dt} + x''y''F_1,$$

$$N_1 = F_{yy} - \frac{dN}{dt} + x''^2 F_1,$$

whence

$$(20) \qquad L_1 x' + M_1 y' = \tfrac{1}{2} T'y', \qquad M_1 x' + N_1 y' = -\tfrac{1}{2} T'x'.$$

If the " δ-process " is applied to the first form of $\delta_1 F$, this form becomes, after reduction,

$$(21) \qquad \delta(\delta_1 F) = T\delta w - w\frac{d}{dt}(F_1 w') + L_1 \delta x^2 + 2M_1 \delta x \delta y + N_1 \delta y^2.$$

§ 7. *Second form of* $\delta(\delta_1 F)$.

In order to apply more conveniently the subscript notation we will place

$$F_1 = H.$$

From the definition (14a) of H follow the equations:

$$F_{x'x'x} = y'^2 H_x, \qquad F_{x'y'x} = -x'y'H_x, \qquad F_{y'y'x} = x'^2 H_x,$$
$$F_{x'x'y} = y'^2 H_y, \qquad F_{x'y'y} = -x'y'H_y, \qquad F_{y'y'y} = x'^2 H_y.$$

Futhermore H is positively homogeneous of degree -3 in x' and y', i. e.,

$$H_{x'} x' + H_{y'} y' = -3H.$$

Making use of these relations and applying the " δ-process " to the second form of $\delta_1 F$ in (18), we obtain

$$(22) \qquad \delta(\delta_1 F) = w\left[A\delta x + B\delta y - \frac{d}{dt}(w'H)\right] + T\delta w,$$

where

$$(23) \qquad A = T_x + \frac{d}{dt}(Hy''), \qquad B = T_y - \frac{d}{dt}(Hx'').$$

A comparison of (21) and (22) then gives

$$(24) \qquad L_1 \delta x^2 + 2M_1 \delta x \delta y + N_1 \delta y^2 = w(A\delta x + B\delta y).$$

§ 8. *The absolute invariant* $\Phi(x, y, x', y', x'', y'', x''', y''', \delta x, \delta y)$.

The absolute invariant $\delta(\delta_1 F)$ contains the variations δx, δy, $\delta x'$, $\delta y'$, $\delta x''$, $\delta y''$. The next step is to split $\delta(\delta_1 F)$ into an aggregate of absolute invariants, one of which contains only the variations δx and δy, and these homogeneously. Then by § 3 we shall derive an invariant with only one set of variables, i. e., not containing any δ's.

For this purpose we first introduce in $\delta(\delta_1 F)$ of § 7 the absolute invariant ω defined by the equation $\omega = wH^{\frac{1}{2}}$, and assume $H \neq 0$ along the curve considered. It follows after reduction that

$$\begin{aligned}\delta(\delta_1 F) = \frac{T}{H^{\frac{1}{2}}}\delta\omega - \frac{T}{2H^{\frac{1}{2}}}\omega\frac{d}{dt}\left(\frac{H_{x'}\delta x + H_{y'}\delta y}{H}\right) - \omega\omega''\\ - \frac{T}{2H^{\frac{1}{2}}}\omega\left[\frac{H_x\delta x + H_y\delta y}{H} - \delta x\frac{d}{dt}\left(\frac{H_{x'}}{H}\right) - \delta y\frac{d}{dt}\left(\frac{H_{y'}}{H}\right)\right]\\ + \frac{\omega}{H^{\frac{1}{2}}}(A\delta x + B\delta y) - \omega^2\left(\frac{H'^2}{4H^2} - \frac{H''}{2H}\right).\end{aligned} \tag{25}$$

Since T and H are invariants of index -1 and $+2$ respectively, and since ω, and therefore ω'' and $\delta\omega$ are absolute invariants, it can be shown that each of the first three terms of $\delta(\delta_1 F)$ is itself an absolute invariant.

On account of the fact that $\delta(\delta_1 F)$ is an absolute invariant itself, we obtain the following absolute invariant under the extended group (1), (3), which contains only the variations δx and δy, and these *linearly* and *homogeneously:*

$$\begin{aligned}&\Phi(x, y, x', y', x'', y'', x''', y''', \delta x, \delta y)\\ &= \frac{1}{2}\frac{T}{H^{\frac{1}{2}}}\left[-\frac{H_x\delta x + H_y\delta y}{H} + \delta x\frac{d}{dt}\left(\frac{H_{x'}}{H}\right) + \delta y\frac{d}{dt}\left(\frac{H_{y'}}{H}\right)\right]\\ &\qquad + \frac{1}{H^{\frac{1}{2}}}(A\delta x + B\delta y) - \omega\left(\frac{H'^2}{4H^2} - \frac{H''}{2H}\right).\end{aligned} \tag{26a}$$

With the help of (24) this may be written

$$\begin{aligned}&\omega\Phi(x, y, x', y', x'', y'', x''', y''', \delta x, \delta y)\\ &= \frac{1}{2}\frac{T\omega}{H^{\frac{1}{2}}}\left[-\frac{H_x\delta x + H_y\delta y}{H} + \delta x\frac{d}{dt}\left(\frac{H_{x'}}{H}\right) + \delta y\frac{d}{dt}\left(\frac{H_{y'}}{H}\right)\right]\\ &\qquad + L_1\delta x^2 + 2M_1\delta x\delta y + N_1\delta y^2 - \omega^2\left(\frac{H'^2}{4H^2} - \frac{H''}{2H}\right).\end{aligned} \tag{26b}$$

§ 9. *The absolute invariant $K(x, y, x', y', x'', y'', x''', y''')$.*

The absolute invariant Φ is homogeneously linear in δx, δy, these latter variables being cogredient with x', y'. Therefore we obtain by § 3 an invariant of index 1, if in Φ we replace δx and δy by $F_{y'}$ and $-F_{x'}$ respectively. In order to obtain an absolute invariant, we divide by $H^{\frac{1}{2}}$ and denote the final result by $-KF$. We have then the following theorem:

THEOREM. *The function $F(x, y, x', y')$ possesses with respect to the point transformation*

$$x = X(u, v), \qquad y = Y(u, v),$$

and its extensions, the following absolute invariant:

$$(27a)\quad \begin{aligned} K(x, y, x', y', x'', y'', x''', y''') &= -\frac{1}{FH}(AF_{y'} - BF_{x'}) + \left(\frac{H'^2}{4H^2} - \frac{1}{2}\frac{H''}{H}\right) \\ &- \frac{1}{2}\frac{T}{FH}\left[-F_{y'}\left\{\frac{H_x}{H} - \frac{d}{dt}\left(\frac{H_{x'}}{H}\right)\right\} + F_{x'}\left\{\frac{H_y}{H} - \frac{d}{dt}\left(\frac{H_{y'}}{H}\right)\right\}\right]. \end{aligned}$$

This invariant may also be written in the following form:

$$(27b)\quad \begin{aligned} K(x, y, x', y', x'', y'', x''', y''') &= -\frac{1}{F^2H}[L_1F_{y'}^2 - 2M_1F_{x'}F_{y'} + N_1F_{x'}^2] \\ &+ \left(\frac{H'^2}{4H^2} - \frac{1}{2}\frac{H''}{H}\right) - \frac{1}{2}\frac{T}{FH}\left[-F_{y'}\left\{\frac{H_x}{H} - \frac{d}{dt}\left(\frac{H_{x'}}{H}\right)\right\} + F_{x'}\left\{\frac{H_y}{H} - \frac{d}{dt}\left(\frac{H_{y'}}{H}\right)\right\}\right]. \end{aligned}$$

Up to this point the curve

$$x = x(t), \qquad y = y(t),$$

for which K is computed and to which the differentiations with respect to t refer, has been entirely arbitrary. K will simplify considerably when this curve is an extremal. In this case $T = 0$, and as a result the right hand members of (20) are zero, and hence we may write with WEIERSTRASS

$$L_1 = y'^2 F_2, \qquad M_1 = -x'y'F_2, \qquad N_1 = x'^2 F_2.$$

The substitution of these in (27*b*) leads to the following result:

Corollary. For the special case of an extremal the invariant K takes the form

$$(28a)\qquad \bar{K} = \frac{1}{4}\frac{F_1'^2}{F_1^2} - \frac{1}{2}\frac{F_1''}{F_1} - \frac{F_2}{F_1},$$

where F_1, F_2 have the same meaning as in the WEIERSTRASS *theory, and the stroke indicates, here and hereafter, that K refers to an extremal.*

Had we used (27*a*), $\bar{K}$ would appear in the form

$$(28b)\qquad \bar{K} = \frac{1}{4}\frac{F_1'^2}{F_1^2} - \frac{1}{2}\frac{F_1''}{F_1} - \frac{\bar{A}F_{y'} - \bar{B}F_{x'}}{FF_1}.$$

By comparing these last two equations a new expression for the WEIERSTRASS function F_2 is found, viz.:

$$(29)\qquad F_2 = \frac{1}{F}(F_{y'}\bar{A} - F_{x'}\bar{B}),$$

where $\bar{A}$ and $\bar{B}$ are the values of A, B along an extremal.

CHAPTER III. *Invariants under combined parameter and extended point transformations. A normal form of $\delta^2 I$.*

§ 10. *Invariants with respect to parameter transformations.*

The principal object of the present chapter is to derive from the invariant K another invariant K_0, which will remain invariant not only under every point transformation but also under every parameter transformation. Let

$$(30) \qquad t = \chi(\tau),$$

be an admissible parameter transformation,* and $\tau = \theta(t)$ the inverse transformation. For brevity set

$$\frac{d\tau}{dt} = \theta'(t) = \lambda,$$

where $\lambda > 0$, and denote by $\bar{\phi}$ the transform of a function $\phi(t)$ by means of (30). We are easily led to the following results:

$$F = \lambda \bar{F}, \qquad F dt = \bar{F} d\tau, \qquad H = \lambda^{-3} \bar{H}, \qquad T = \bar{T}.$$

We define $I(x, y, x', y', x'', y'', \cdots)$ as an absolute invariant with respect to the parameter transformation (30), if it satisfies

$$I(x, y, x', y', \cdots) = I(\bar{x}, \bar{y}, \bar{x}', \bar{y}', \cdots),$$

or $I = \bar{I}$, for all systems of the arguments connected by the relations:

$$\begin{array}{ll} x = \bar{x}, & y = \bar{y}, \\ x' = \lambda \bar{x}', & y = \lambda \bar{y}', \\ x'' = \lambda^2 \bar{x}'' + \lambda' \bar{x}', & y = \lambda^2 \bar{y}'' + \lambda' \bar{y}', \\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot & \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot \end{array}$$

where $\lambda > 0$.

If the functions $x(t)$, $y(t)$ depend upon a parameter ϵ, whereas $\chi(\tau)$ of (30) is independent of ϵ, one finds at once that

$$\delta x = \delta \bar{x}, \qquad \delta y = \delta \bar{y},$$

where the operator δ is again equivalent to $\epsilon \cdot \partial / \partial \epsilon$.

Then more generally, if $I(x, y, x', y', \cdots)$ be an absolute invariant with respect to parameter transformation, the equation $I = \bar{I}$ becomes an identity in ϵ and τ, if we substitute in the left hand side $t = \chi(\tau)$. Hence

$$\delta I = \delta \bar{I};$$

i. e., the "δ-process" transforms every absolute invariant with respect to parameter transformations again into an absolute invariant.

* BOLZA, loc. cit., § 24*a*.

It follows that

$$w = \lambda \bar{w}, \qquad \omega = \lambda^{-\frac{1}{2}} \bar{\omega},$$

where

$$\omega = w H^{\frac{1}{2}}.$$

§ 11. *Combined point and parameter transformation.*

To a curve

$$L: \qquad x = x(t), \qquad y = y(t),$$

we now apply first the point transformation $(1a)$ and then the parameter transformation (30), or vice versa, since they are commutable. As a result of the relations found in § 2 and § 10, we find

$$F(x, y, x', y') = G(u, v, u', v') = \lambda G(\bar{u}, \bar{v}, \bar{u}', \bar{v}'),$$

$$F_1(x, y, x', y') = D^{-2} G_1(u, v, u', v') = D^{-2} \lambda^{-3} G_1(\bar{u}, \bar{v}, \bar{u}', \bar{v}'),$$

$$T_F(x, y, x', y', x'', y'') = D^{-1} T_G(u, v, u', v', u'', v'') = D^{-1} T_G(\bar{u}, \bar{v}, \bar{u}', \bar{v}', \bar{u}'', \bar{v}''),$$

$$\omega = \lambda^{-\frac{1}{2}} \bar{\omega},$$

where

$$\bar{\omega} = (\bar{v}' \delta \bar{u} - \bar{u}' \delta \bar{v}) \cdot \{ H(u, v, \bar{u}', \bar{v}') \}^{\frac{1}{2}}.$$

From these last results it follows that the functions

$$(31) \qquad S = \frac{T}{H^{\frac{1}{2}} F^{\frac{3}{2}}},$$

$$(32) \qquad V = \omega F^{\frac{1}{2}},$$

are absolute invariants under the application of combined point and parameter transformations.

§ 12. *The absolute invariant* $\Phi_0(x, y, x', y', x'', y'', x''', y''', \delta x, \delta y)$.

Our starting point is the fact that wT/F is an absolute invariant under the combined group, and that when the "δ-process" is applied to this expression, we again obtain an absolute invariant which in turn may be broken up into several separate absolute invariants. We may write

$$\delta\left(\frac{wT}{F}\right) = \delta(VS) = S\delta V + V\delta S,$$

and will fix our attention on δS, which is an absolute invariant since S, V, and δV are absolute invariants. This has the form

$$\delta S = \delta\left(\frac{T}{H^{\frac{1}{2}} F^{\frac{3}{2}}}\right) = S\left(\frac{\delta T}{T} - \frac{1}{2}\frac{\delta H}{H} - \frac{3}{2}\frac{\delta F}{F}\right).$$

Since S is an absolute invariant and since λ is independent of ϵ, it follows that $\delta F/F$ is an absolute invariant, and we are led to the following expression as an absolute invariant under the combined group :

$$\frac{\delta T}{T}-\frac{1}{2}\frac{\delta H}{H}=\frac{1}{T}\left[A\delta x+B\delta y-H^{\frac{1}{3}}\left\{\omega''+\omega\left(\frac{H'^2}{4H^2}-\frac{1}{2}\frac{H''}{H}\right)\right\}\right]$$
$$-\frac{1}{2}\left[\frac{H_x\delta x+H_y\delta y}{H}+\frac{H_{x'}\delta x'+H_{y'}\delta y'}{H}\right]$$

Next we notice that if I is an absolute invariant under the combined group then so is

$$\frac{1}{F}\frac{dI}{dt}.$$

Setting

$$\text{(33)}\qquad \frac{1}{F}V'=V_1,\qquad \frac{1}{F}V_1'=V_2,$$

in which V_1 and V_2 are each absolute invariants, we may write

$$\frac{\delta T}{T}-\frac{\delta H}{2H}=\frac{1}{T}(A\delta x+B\delta y)+\frac{V}{2SF^2}\left[\frac{F'''}{F}-\frac{3}{2}\frac{F''^2}{F^2}\right]$$
$$-\frac{1}{F^2}\frac{V}{S}\left[\frac{H'^2}{4H^2}-\frac{1}{2}\frac{H''}{H}\right]+\frac{1}{2}\frac{\delta x}{F}\frac{d}{dt}\left[\frac{FH_{x'}}{H}\right]-\frac{1}{2}\left[\frac{H_x\delta x+H_y\delta y}{H}\right]$$
$$+\frac{1}{2}\frac{\delta y}{F}\frac{d}{dt}\left[\frac{FH_{y'}}{H}\right]-\frac{V_2}{S}-\frac{1}{2F}\frac{d}{dt}\left[\frac{F(H_{x'}\delta x+H_{y'}\delta y)}{H}\right].$$

Since H is homogeneous in x', y' of order 3, it follows that the last term and also the next to the last term are absolute invariants. Hence the remaining terms multiplied by S, constitute an absolute invariant under the combined group. Denoting it by Φ_0 we obtain

$$\Phi_0(x,y,x',y',x'',y'',x''',y''',\delta x,\delta y)=P\delta x+Q\delta y$$
$$+\frac{V}{2F^2}\left[\frac{F'''}{F}-\frac{3}{2}\frac{F''^2}{F^2}\right]-\frac{V}{F^2}\left[\frac{H'^2}{4H^2}-\frac{1}{2}\frac{H''}{H}\right],$$

in which

$$P=\frac{SA}{T}-\frac{S}{2}\frac{H_x}{H}+\frac{S'}{2F}\frac{d}{dt}\left[\frac{FH_{x'}}{H}\right],$$

$$Q=\frac{SB}{T}-\frac{S}{2}\frac{H_y}{H}+\frac{S}{2F}\frac{d}{dt}\left[\frac{FH_{y'}}{H}\right].$$

A and B have the meanings given in (23), and S is defined by (31).

§ 13. *The absolute invariant* $K_0(x, y, x', y', x'', y'', x''', y''')$.

The absolute invariant Φ_0 under the combined group is homogeneous and linear in δx and δy, and these variables are cogredient with x', y' Since the parameter part of the group introduces only powers of λ, and since in the cogredient variables no such factors are possible, we can apply § 3 and get an invariant of index 1 if in Φ_0 we replace δx and δy by $F_{y'}$ and $-F_{x'}$ respectively. In order to obtain an absolute invariant we multiply by $1/H^{\frac{1}{2}}F^{\frac{3}{2}}$ and denote the result by $-K_0$. Noticing that $V = \omega F^{\frac{1}{2}}$ becomes by this substitution $F^{\frac{3}{2}}H^{\frac{1}{2}}$, we obtain thus the theorem:

THEOREM. *The function* $F(x, y, x', y')$ *possesses with respect to the extended point transformation*

$$x = X(u, v), \qquad y = Y(u, v),$$

and the parameter transformation $t = \chi(\tau)$ *the following absolute invariant:*

$$K_0(x, y, x', y', x'', y'', x''', y''')$$

$$(34) \qquad = \frac{1}{F^2}\left[K(x, y, x', y', x'', y'', x''', y''') - \frac{1}{2}\left(\frac{F''}{F} - \frac{3}{2}\frac{F'^2}{F^2}\right)\right] + \frac{T}{2HF^4}\left[\frac{H_{y'}F_{x'} - H_{x'}F_{y'}}{H}\right]F',$$

where K is given by $(27a)$.

As before in § 9, so here in this chapter we have used as yet a general curve

$$x = x(t), \qquad y = y(t),$$

for which K_0 has been computed, and to which the differentiation with respect to t referred. In case the curve is an extremal along which $T = 0$ the formula (34) becomes

$$(35) \qquad \overline{K}_0 = \frac{1}{F^2}\left[\frac{1}{4}\frac{F_1'^2}{F_1^2} - \frac{1}{2}\frac{F_1''}{F_1} - \frac{F_2}{F_1} - \frac{1}{2}\frac{F''}{F} + \frac{3}{4}\frac{F'^2}{F^2}\right].$$

§ 14. *Computation of* $\overline{K}_0$ *for Kneser's curvilinear coördinates.*

We suppose that the u, v introduced by $(1a)$ instead of x, y are KNESER's curvilinear coördinates;* in other words, the curves $v = c$ are extremals for the integral

$$I' = \int_{t_0}^{t_1} G(u, v, u', v')\,dt,$$

while the curves $u = c$ are the transversals of this set of extremals. The function $G(u, v, u', v')$ has the following characteristic properties:†

$$G(u, v, u', 0) = u', \qquad G_{u'}(u, v, u', 0) = 1, \qquad G_{v'}(u, v, u', 0) = 0.$$

* KNESER, loc. cit., § 16; BOLZA, loc. cit., § 35 *b*, *c*.

† BOLZA, loc. cit., p. 185.

We propose to compute $\bar{K}_0$. The arguments of the functions G, G_1, G_2 are

$$u = u(t), \qquad v = c, \qquad u' = u'(t), \qquad v' = v'(t) = 0,$$

and one can easily compute the values

$$L = 0, \qquad M = 0, \qquad N = -uu''G_1.$$

But since $\bar{K}_0$ is an absolute invariant under the combined group, we may choose the parameter t on the extremal $v = c$ at pleasure. We select $u = \pm t$ according as u increases or decreases with t. Then also

$$N = 0,$$

and from (19)

$$L_1 = M_1 = N_1 = 0,$$

so that

$$G_2 = 0.$$

Further

$$G = u' = \pm 1, \qquad G' = 0, \qquad G'' = 0.$$

Finally if we put

$$G_1(u, v, \pm 1, 0) = h(u, v),$$

then

$$G_1' = \pm h_u, \qquad G_1'' = \pm h_{uu},$$

and substituting these values in (35) we have the following theorem:

THEOREM. *For* KNESER's *normal form of the integral I the absolute invariant $\bar{K}_0$ takes the following form,*

$$\bar{K}_0 = -\frac{1}{\sqrt{h}} \frac{\partial^2 \sqrt{h}}{\partial u^2}, \tag{36}$$

where $h(u, v) = G_1(u, v, \pm 1, 0)$.

§ 15. *An invariantive normal form of the second variation.*

We restrict now our problem by supposing the end points to be fixed. Then the second variation computed along an extremal of

$$I = \int_{t_0}^{t_1} F(x, y, x', y')\, dt$$

is

$$\overline{\delta^2 I} = \int_{t_0}^{t_1} w \delta T dt.$$

By using the results of §§ 7, 8, 9 this may be transformed into

$$\overline{\delta^2 I} = -\int_{t_0}^{t_1} \omega(\omega'' + \bar{K}\omega)\, dt,$$

and then, by introducing V as defined in (32), into

$$\overline{\delta^2 I} = -\int_{t_0}^{t_1} F V(V_2 + \overline{K}_0 V)\,dt.$$

In case a change of variable is made by means of

$$\alpha = \int_{t_0}^{t} F dt,$$

where to t_0 and t_1 correspond α_0 and α_1, we find

$$\overline{\delta^2 I} = -\int_{\alpha_0}^{\alpha_1} \left(\frac{d^2 V}{d\alpha^2} + \overline{K}_0 V \right) V d\alpha.$$

Integrating the first term by parts, and recalling that V by definition vanishes for α_0 and α_1, we have the result:

When F_1 is positive along the extremal under consideration, the second variation of

$$I = \int_{t_0}^{t_1} F(x, y, x', y')\,dt$$

may be reduced to the invariantive normal form

$$\delta^2 I = \int_{\alpha_0}^{\alpha_1} \left[\left(\frac{dV}{d\alpha} \right)^2 - \overline{K}_0 V^2 \right] d\alpha, \tag{37}$$

in which

$$V = \omega F^{\frac{1}{2}}, \qquad \alpha = \int_{t_0}^{t} F dt,$$

and $\overline{K}_0$ is the invariant defined in § 13.

As a result of this form the following corollary may at once be stated:

Corollary. In case $\overline{K}_0 < 0$ along the extremal we have $\delta^2 I > 0$.

CHAPTER IV. *Application to the geodesic problem.*

§ 16.

The function $F(x, y, x', y')$ for the geodesic problem has the form

$$F(x, y, x', y') = \sqrt{\mathcal{E} x'^2 + 2\mathcal{F} x'y' + \mathcal{G} y'^2},$$

where $\mathcal{E}$, $\mathcal{F}$, $\mathcal{G}$ are the well-known functions of x and y only, used in surface theory. The functions H, F_1 and T have the following values:

$$F_1 = \frac{\mathcal{E}\mathcal{G} - \mathcal{F}^2}{\left(\sqrt{\mathcal{E} x'^2 + 2\mathcal{F} x'y' + \mathcal{G} y'^2}\right)^3}, \qquad T = \frac{\Gamma}{\left(\sqrt{\mathcal{E} x'^2 + 2\mathcal{F} x'y' + \mathcal{G} y'^2}\right)^3}, \tag{38}$$

where

$$\Gamma = (\mathcal{E}\mathcal{G} - \mathcal{F}^2)(x'y'' - x''y')$$
$$+ (\mathcal{E}x' + \mathcal{F}y')[(\mathcal{F}_x - \tfrac{1}{2}\mathcal{E}_y)x' + \mathcal{G}_x x'y' + \tfrac{1}{2}\mathcal{G}_y y'^2]$$
$$- (\mathcal{F}x' + \mathcal{G}y')[\tfrac{1}{2}\mathcal{E}_x x'^2 + \mathcal{E}_y x'y' + (\mathcal{F}_y - \tfrac{1}{2}\mathcal{G}_x)y'^2].$$

a) The invariant S.

In the case of a geodesic the absolute invariant

$$S = \frac{T}{H^{\frac{1}{2}}F^{\frac{3}{2}}}$$

is identical with the geodesic curvature.*

b) The invariant $\overline{K}_0$.

Since $\overline{K}_0$ is an absolute invariant for point as well as parameter transformations, we select the Gaussian normal form

$$ds^2 = du^2 + m^2 dv^2,$$

in which u, v are geodesic parallel coördinates, and take for t the arc of the curve on the surface, so that

$$G = \sqrt{u'^2 + m^2 v'^2} = 1.$$

Since this system of coördinates is identical for the geodesic problem with the KNESER coördinates in § 14, we may use the results there found.

From (38) it follows that

$$G_1 = m^2,$$

and by using (36), that

$$\overline{K}_0 = -\frac{1}{m}\frac{\partial^2 m}{\partial u^2}.$$

Accordingly we have the result:

In the case of the geodesic problem the absolute invariant $\overline{K}_0$ is identical with the Gaussian curvature.†

Combining this result with (37), we have the JACOBI-BONNET ‡ theorem:

On a surface of negative curvature a given point has no conjugate point on the geodesics which pass through it.

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* BOLZA, loc. cit., pp. 129, 146.

† SCHEFFERS, *Anwendung der Differential- und Integral-Rechnung auf Geometrie*, vol. 2, p. 503.

‡ Comptes Rendus, vol. 40 (1855), p. 1311; vol. 41 (1855), p. 32. JACOBI, *Gesammelte Werke*, supplementary volume, p. 46.

c) Computation of K_0.

For the computation of K_0 we assume a system of isometric coördinates; i. e.,

$$F = \sqrt{\mathcal{E}(x'^2 + y'^2)}$$

and substitute this value of F in (34). Since (34) is an invariant under parameter transformation, we may then choose as the parameter t, the arc s of the curve which makes $F = 1$. It will be found that

$$K_0 = -\frac{\mathcal{E}_{xx} + \mathcal{E}_{yy}}{2\mathcal{E}^2} + \frac{\mathcal{E}_x + \mathcal{E}_y}{2\mathcal{E}^3} - \frac{1}{2}\frac{T^2}{\mathcal{E}^2}. \tag{39}$$

In the case of the geodesics the absolute invariant K_0 has the value

$$K_0 = \frac{1}{\rho} - \frac{1}{2}\left[\frac{1}{\rho_g}\right]^2, \tag{40}$$

where $1/\rho$ is the Gaussian curvature * of the surface at the point (x, y) and $1/\rho_g$ is the geodesic curvature † at the same point.

* SCHEFFERS, loc. cit., vol. 2, p. 493.

† Cf. § 16 a), above.